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On non-Abelian holonomies

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Abstract
We provide a method and results for the calculation of the holonomy of a Yang–Mills connection in an arbitrary triangular path, in an expansion (developed here to fifth order) in powers of the corresponding segments. The results might have applications in generalizing to Yang–Mills fields previous calculations of the corrections to particle dynamics induced by loop quantum gravity, as well as in the field of random lattices.

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1. Introduction

Constraints in Lorentz covariance violations have been experimentally studied for a long time [1] by obtaining observational bounds upon the violating parameters. Recent experiments have shown an impressive increase in their sensitivities, thus producing even more stringent bounds [2].

In order to correlate such experimental results, Kostelecky and collaborators have proposed a phenomenological extension of the standard model, which incorporates the most general Lorentz-violating interactions compatible with power counting renormalizability together with the particle structure of the standard model [3]. An impressive number of applications to very different processes have already been considered, as can be seen in [4], for example.

Different models accounting for minute Lorentz violations have recently arisen in the context of quantum gravity induced corrections to the propagation and interactions of particles [5–9]. This amounts to realizing the generic violating parameters appearing in the standard model extension in terms of specific quantities involving the Planck length ℓ_{P} together with additional physically relevant objects. Moreover, the high precision obtained in the determination of the experimental bounds has brought quantum gravity induced effects to the level of observable phenomena [10–13].

On the other hand, in the early 1980s, the growing importance of computer simulations of gauge theories required a short distance cutoff of geometrical origin such as a lattice. However, regular lattices break essential symmetries of continuum theories such as translational and rotational invariance. Motivated by the need to maintain these symmetries, the field theory on a random lattice was suggested [14]. Later on, the connection of random lattices with quantum gravity and strings was studied and low-dimensional systems on random lattices were solved using matrix model techniques [15].

In this work we concentrate on some aspects arising in the process of generalizing the loop quantum gravity inspired model described in [6, 7] to Yang–Mills fields, in order to obtain the non-Abelian generalization of the corrections previously found for the dynamics of photons. Namely, corrections to standard matter dynamics are obtained by means of calculating non-Abelian holonomies, either of gravitational or Yang–Mills-type, around triangular paths. To this end we have to revise and extend the procedure of [16] that was applied to the case of rectangular cells. It turns out that the method we present in this paper contains as a particular case the result of [16], though it is applicable to arbitrary cells made up of triangles. The basic building block in our analysis is the holonomy along a straight line segment, whose characteristic property of being path-ordered is consistently maintained in all orders in our expansion.

The problem we deal with here is closely related to the non-Abelian Stokes theorem which has been repeatedly discussed in the literature [16, 17].

This paper is organized as follows: in section 2 we state the problem to be dealt with and introduce some notation. Section 3 summarizes the results for the Abelian case which we intend to generalize here. The non-Abelian case is subsequently discussed in section 4 which contains our main results. Using the procedure of [16] the corresponding calculation is performed in section 5, which allows us to show some discrepancies that arise between the two methods. Finally we close with a summary and discussion in section 6.

The computations involved in deriving $\mathbf{h}^{(5)}_{\alpha_{IJ}}$ were done using FORM [18].

2. Statement of the problem

The proposed method to obtain the quantum gravity induced corrections to the Yang–Mills Lagrangian requires the calculation of the object

$$T_\rho = \mathrm{tr}(\mathbf{G}_\rho \mathbf{h}_{\alpha_{IJ}}) \tag{1}$$

where $\mathbf{G}_\rho$ are the generators of the corresponding Lie algebra and $\mathbf{h}_{\alpha_{IJ}}$ is the holonomy of the Yang–Mills connection $\mathbf{A}_a = A^\rho_a \mathbf{G}_\rho$ in the triangle α_{IJ}, with vertex v, defined by the vectors $\vec{s}_I$ and $\vec{s}_J$, arising from the vertex v in the way described below.

Our main task will be to construct an expansion of T_ρ in powers of the segments s^a_I and s^b_J.

To be more precise, we have

$$\mathbf{h}_{\alpha_{IJ}} = P \exp\left(\oint_{\alpha_{IJ}} \mathbf{A}_a(\vec{x}(s)) \frac{\mathrm{d}x^a}{\mathrm{d}s}\,\mathrm{d}s\right) \tag{2}$$

where P is a path-ordered product to be specified later. As shown in figure 1, the closed path α_{IJ}, parametrized by $\vec{x}(s)$, is defined in the following way: we start from the vertex v following a straight line with the direction and length of $\vec{s}_I$, then follow another straight line in

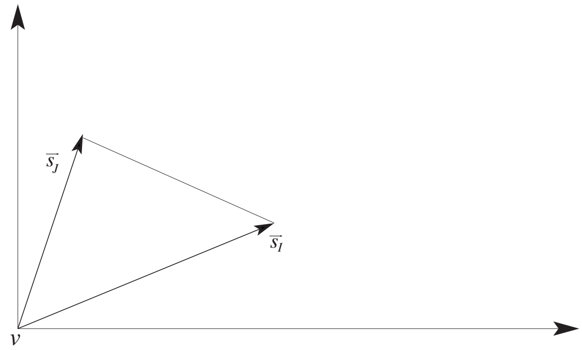

Figure 1. Triangle α_{IJ} with vertex v.

the direction and length of $\vec{s}_J - \vec{s}_I$, and finally return to v following $-\vec{s}_J$. From the definition of the holonomy, we have the transformation property

$$\mathbf{h}_{\alpha_{IJ}} \rightarrow \mathbf{U}(v)\mathbf{h}_{\alpha_{IJ}}\mathbf{U}(v)^{-1} \tag{3}$$

under a gauge transformation of the connection, where $\mathbf{U}(v)$ is a group element evaluated at the vertex v. In other words, $\mathbf{h}_{\alpha_{IJ}}$ transforms covariantly under the group.

3. The Abelian case

The corresponding calculation was performed in [6] and here we summarize the results in order to have the correct expressions to which the non-Abelian result must reduce when taking the commuting limit. In this case equation (1) reduces to

$$T = \exp(\Phi_{IJ}) - 1 \tag{4}$$

where Φ_{IJ} is the magnetic flux through the area of the triangle, given by

$$\begin{aligned}\Phi^B(F_{IJ}) &= \oint_{\alpha_{IJ}} \mathrm{d}t\, \dot{s}^a(t) A_a(t) \\ &= \int_{\vec{v}}^{\vec{v}+\vec{s}_I} A_a\, \mathrm{d}x^a + \int_{\vec{v}+\vec{s}_I}^{\vec{v}+\vec{s}_J} A_a\, \mathrm{d}x^a + \int_{\vec{v}+\vec{s}_J}^{\vec{v}} A_a\, \mathrm{d}x^a \end{aligned}\tag{5}$$

where the connection $A_a(\vec{x}(s))$ is now a commuting object.

The basic building block in (5) is

$$\begin{aligned}\int_{\vec{v}_1}^{\vec{v}_2} A_a(\vec{x})\, \mathrm{d}x^a &= \int_0^1 A_a(\vec{v}_1 + t(\vec{v}_2 - \vec{v}_1))(\vec{v}_2 - \vec{v}_1)^a\, \mathrm{d}t \\ &= \int_0^1 A_a(\vec{v}_1 + t\vec{\Delta})\Delta^a\, \mathrm{d}t \\ &= \left(1 + \frac{1}{2!}\Delta^b\partial_b + \frac{1}{3!}(\Delta^b\partial_b)^2 + \cdots\right)\Delta^a A_a(v)\end{aligned}\tag{6}$$

with $\Delta^a = (\vec{v}_2 - \vec{v}_1)^a$. The infinite series in parenthesis is

$$F(x) = 1 + \frac{1}{2!}x + \frac{1}{3!}x^2 + \frac{1}{4!}x^3 + \cdots = \frac{\mathrm{e}^x - 1}{x} \tag{7}$$

yielding

$$\int_{\vec{v}_1}^{\vec{v}_2} A_a(\vec{x})\, \mathrm{d}x^a = F(\Delta^a\partial_a)(\Delta^a A_a(\vec{v}_1)). \tag{8}$$

In the following we employ the notation $\Delta^a V_a = \vec{\Delta} \cdot \vec{v}$. Using the above result in the three integrals appearing in (5) and after some algebra, we obtain

$$\Phi^B(F_{IJ}) = F_1(\vec{s}_I \cdot \nabla, \vec{s}_J \cdot \nabla) s_J^a s_I^b (\partial_a A_b(\vec{v}) - \partial_b A_a(\vec{v}))$$
$$= F_1(\vec{s}_I \cdot \nabla, \vec{s}_J \cdot \nabla) s_J^a s_I^b \epsilon_{abc} B^c(v) \tag{9}$$

where the gradient acts upon the coordinates of $\vec{v}$. The function F_1 is

$$F_1(x, y) = \frac{y(\mathrm{e}^x - 1) - x(\mathrm{e}^y - 1)}{xy(y - x)} = -\sum_{n=1}^{\infty} \frac{1}{(n+1)!} \frac{x^n - y^n}{x - y}. \tag{10}$$

Let us emphasize that $F_1(x, y)$ is just a power series in the variables x and y. Expanding in powers of the segments s_I^a we obtain

$$\Phi^B(F_{IJ}) = \left(1 + \tfrac{1}{3}\left(s_I^c + s_J^c\right)\partial_c + \tfrac{1}{12}\left(s_I^c s_I^d + s_I^c s_J^d + s_J^c s_J^d\right)\partial_c \partial_d + \cdots\right) \tfrac{1}{2} s_I^a s_J^b \epsilon_{abc} B^c(v). \tag{11}$$

Note that the combination

$$\tfrac{1}{2} s_I^a s_J^b \epsilon_{abc} = \mathcal{A} n_c \tag{12}$$

is just the oriented area of the triangle with vertex v and sides s_I^c, s_J^c, joining at this vertex, having value $\mathcal{A}$ and unit normal vector n_c.

To conclude we have to calculate

$$(\exp(\Phi^B(F_{IJ})) - 1) = \sum_{n=2}^{\infty} \frac{1}{n!} (\Phi^B(F_{IJ}))^n = \sum_{n=2}^{\infty} M_{nIJ} \tag{13}$$

where the subindex n labels the corresponding power in the vectors s^a. The results are

$$M_{2IJ} := s_I^a s_J^b \frac{1}{2!} F_{ab} \tag{14}$$

$$M_{3IJ} := s_I^a s_J^b \frac{1}{3!} (x_I + x_J) F_{ab} \tag{15}$$

$$M_{4IJ} := s_I^a s_J^b \frac{1}{4!} \left(x_I^2 + x_I x_J + x_J^2\right) F_{ab} + s_I^a s_J^b s_I^c s_J^d \tfrac{1}{8} F_{ab} F_{cd} \tag{16}$$

$$M_{5IJ} := \frac{1}{5!} \left(x_I^3 + x_J^3 + x_I^2 x_J + x_I x_J^2\right) s_I^a s_J^b F_{ab} + \frac{s_I^a s_J^b s_I^c s_J^d}{4!} [(x_I + x_J) F_{ab} F_{cd} + F_{ab} (x_I + x_J) F_{cd}] \tag{17}$$

up to fifth order. We are using the notation $x_I = \vec{s}_I \cdot \nabla = s_I^a \partial_a$.

We expect that the non-Abelian generalization of the quantities (14)–(17) is produced by the replacement

$$A_a \to \mathbf{A}_a = A_a^\rho G_\rho \qquad \partial_a \to \mathbf{D}_a = \partial_a - [\mathbf{A}_a,] \tag{18}$$

$$F_{ab} \to \mathbf{F}_{ab} = \partial_a \mathbf{A}_b - \partial_b \mathbf{A}_a - [\mathbf{A}_a, \mathbf{A}_b]. \tag{19}$$

Nevertheless, at this level there are potential ordering ambiguities which will be resolved in the following sections.

4. The non-Abelian case

In a way similar to the Abelian case we separate the calculation of the holonomy $\mathbf{h}_{\alpha_{IJ}}$ into three basic pieces through the straight lines along the sides of the triangle α_{IJ}. We have

$$\mathbf{h}_{\alpha_{IJ}} = P(\mathrm{e}^{L_3}) P(\mathrm{e}^{L_2}) P(\mathrm{e}^{L_1}) \equiv U_3 U_2 U_1 \tag{20}$$

where

$$L_1 = \int_0^1 \mathrm{d}t\, \mathbf{A}_a(\vec{v} + t\vec{s}_I)s_I^a \tag{21}$$

$$L_2 = \int_0^1 \mathrm{d}t\, \mathbf{A}_a(\vec{v} + \vec{s}_I + t(\vec{s}_J - \vec{s}_I))\big(s_J^a - s_I^a\big) \tag{22}$$

$$L_3 = \int_0^1 \mathrm{d}t\, \mathbf{A}_a(\vec{v} + \vec{s}_J - t\vec{s}_J)\big(-s_J^a\big). \tag{23}$$

Here we have parametrized each segment with $0 \leqslant t \leqslant 1$.

4.1. The basic building block

Let us consider in detail the contribution

$$U_1 = P(\mathrm{e}^{L_1}) \qquad L_1 = \int_0^1 \mathrm{d}t\, \mathbf{A}_a(\vec{v} + t\vec{s}_I)s_I^a \tag{24}$$

with $\vec{s}_I = \{s_I^a\}$.

Using the definition

$$U_1 = 1 + \int_0^1 \mathrm{d}t\, \mathbf{A}_a(\vec{v} + t\vec{s}_I)s_I^a + \int_0^1 \mathrm{d}t \int_0^t \mathrm{d}t'\, \mathbf{A}_a(\vec{v} + t\vec{s}_I)\mathbf{A}_b(\vec{v} + t'\vec{s}_I)s_I^a s_I^b$$
$$+ \int_0^1 \mathrm{d}t \int_0^t \mathrm{d}t' \int_0^{t'} \mathrm{d}t''\, \mathbf{A}_a\mathbf{A}_b\mathbf{A}_c s_I^a s_I^b s_I^c + \cdots \tag{25}$$

for the path ordering, we arrive at the following expression:

$$U_1 = 1 + I_1(x)\mathbf{A}_a(v)s_I^a + I_2(x,\bar{x})\mathbf{A}_a(v)\bar{\mathbf{A}}_b(v)s_I^a s_I^b$$
$$+ I_3(x,\bar{x},\bar{\bar{x}})\mathbf{A}_a(v)\bar{\mathbf{A}}_b(v)\bar{\bar{\mathbf{A}}}_c(v)s_I^a s_I^b s_I^c + \cdots . \tag{26}$$

Here we are adopting the conventions

$$x = s_I^c\partial_c \qquad \bar{x} = s_I^c\bar{\partial}_c \qquad \bar{\bar{x}} = s_I^c\bar{\bar{\partial}}_c \tag{27}$$

$$I_1 = F(x) \qquad I_2 = \frac{F(x+\bar{x}) - F(x)}{\bar{x}} \tag{28}$$

$$I_3 = \frac{1}{\bar{\bar{x}}}\left[\frac{1}{\bar{x}+\bar{\bar{x}}}(F(x+\bar{x}+\bar{\bar{x}}) - F(x)) - \frac{1}{\bar{x}}(F(x+\bar{x}) - F(x))\right] \tag{29}$$

with $F(x)$ given by equation (7). The notation in equation (26) is that each operator $x, \bar{x}, \bar{\bar{x}}$ acts only in the corresponding field A, $\bar{A}$, $\bar{\bar{A}}$ respectively. We write

$$U_1 = \sum_N U_1^{(N)} \tag{30}$$

where the superindex N indicates the powers of s_I^a contained in each term. A detailed calculation produces

$$U_1^{(1)} = s_I^a\mathbf{A}_a \qquad U_1^{(2)} = \tfrac{1}{2}\big(x s_I^a\mathbf{A}_a + s_I^a s_I^b\mathbf{A}_a\mathbf{A}_b\big) \tag{31}$$

$$U_1^{(3)} = \frac{1}{3!}\big(x^2 s_I^a\mathbf{A}_a + (\bar{x}+2x)s_I^a s_I^b\mathbf{A}_a\bar{\mathbf{A}}_b + s_I^a s_I^b s_I^c\mathbf{A}_a\mathbf{A}_b\mathbf{A}_c\big) \tag{32}$$

$$U_1^{(4)} = \frac{1}{4!}\big[x^3 s_I^a\mathbf{A}_a + (3x^2+3x\bar{x}+\bar{x}^2)s_I^a s_I^b\mathbf{A}_a\bar{\mathbf{A}}_b + (3x+2\bar{x}+\bar{\bar{x}})s_I^a s_I^b s_I^c\mathbf{A}_a\bar{\mathbf{A}}_b\bar{\bar{\mathbf{A}}}_c$$
$$+ s_I^a s_I^b s_I^c s_I^d\mathbf{A}_a\mathbf{A}_b\mathbf{A}_c\mathbf{A}_d\big]. \tag{33}$$

Specializing to the case $\vec{s}_I = (a, 0, 0)$ and to third order in a we obtain

$$U_1 = 1 + a\mathbf{A}_1 + \tfrac{1}{2}a^2\left(\partial_1\mathbf{A}_1 + \mathbf{A}_1^2\right) + \frac{1}{3!}a^3\left(\partial_1^2\mathbf{A}_1 + \mathbf{A}_1\partial_1\mathbf{A}_1 + 2(\partial_1\mathbf{A}_1)\mathbf{A}_1 + \mathbf{A}_1^3\right). \tag{34}$$

Next we compare our result (34) with the calculation according to the method of [16]. Using equation (3.15a) of this reference for L_1 we obtain

$$\mathrm{e}^{L_1} = 1 + a\mathbf{A}_1 + \tfrac{1}{2}a^2\left(\partial_1\mathbf{A}_1 + \mathbf{A}_1^2\right) + \frac{1}{3!}a^3\left(\partial_1^2\mathbf{A}_1 + \tfrac{3}{2}(\mathbf{A}_1\partial_1\mathbf{A}_1 + (\partial_1\mathbf{A}_1)\mathbf{A}_1) + \mathbf{A}_1^3\right) \tag{35}$$

which does not agree with the expression (34). This basic discrepancy propagates when combining more segments and ultimately it is the source of the differences between our results and those obtained via the application of the methods in [16].

It is interesting to remark that in the Abelian limit both (34) and (35) reduce to

$$U_1 = 1 + aA_1 + \tfrac{1}{2}a^2\left(\partial_1A_1 + A_1^2\right) + \frac{1}{3!}a^3\left(\partial_1^2A_1 + 3A_1\partial_1A_1 + A_1^3\right) \tag{36}$$

which is obtained from a direct calculation using the expression (8).

4.2. *The holonomy* $\mathbf{h}_{\alpha_{IJ}}$

Now we put the remaining pieces together in order to calculate $\mathbf{h}_{\alpha_{IJ}} = U_3U_2U_1$. Using the notation

$$y = s_J^a\partial_a \tag{37}$$

and starting from the basic structure (26) we obtain, *mutatis mutandis*,

$$U_2^{(1)} = \left(s_J^a - s_I^a\right)\mathbf{A}_a \tag{38}$$

$$U_2^{(2)} = \tfrac{1}{2}\left[(x+y)\left(s_J^a - s_I^a\right)\mathbf{A}_a + \left(s_J^a - s_I^a\right)\left(s_J^b - s_I^b\right)\mathbf{A}_a\mathbf{A}_b\right] \tag{39}$$

$$\begin{aligned}U_2^{(3)} = \frac{1}{3!}\Big[&(x^2+y^2+xy)\left(s_J^a - s_I^a\right)\mathbf{A}_a + (x+2y+\bar{y}+2\bar{x})\left(s_J^a - s_I^a\right)\left(s_J^b - s_I^b\right)\mathbf{A}_a\bar{\mathbf{A}}_b\\ &+\left(s_J^a - s_I^a\right)\left(s_J^b - s_I^b\right)\left(s_J^c - s_I^c\right)\mathbf{A}_a\mathbf{A}_b\mathbf{A}_c\Big]\end{aligned} \tag{40}$$

$$\begin{aligned}U_2^{(4)} = \frac{1}{4!}\Big[&(x^3+y^3+x^2y+xy^2)\left(s_J^a - s_I^a\right)\mathbf{A}_a + (x\bar{y}+3x\bar{x}+x^2+2xy\\ &+2\bar{x}\bar{y}+3\bar{x}^2+3y^2+\bar{y}^2+3y\bar{y}+5\bar{x}y)\left(s_J^a - s_I^a\right)\left(s_J^b - s_I^b\right)\mathbf{A}_a\bar{\mathbf{A}}_b\\ &+(x+2\bar{x}+3\bar{\bar{x}}+2\bar{y}+\bar{\bar{y}}+3y)\left(s_J^a - s_I^a\right)\left(s_J^b - s_I^b\right)\left(s_J^c - s_I^c\right)\mathbf{A}_a\bar{\mathbf{A}}_b\bar{\bar{\mathbf{A}}}_c\\ &+\left(s_J^a - s_I^a\right)\left(s_J^b - s_I^b\right)\left(s_J^c - s_I^c\right)\left(s_J^d - s_I^d\right)\mathbf{A}_a\mathbf{A}_b\mathbf{A}_c\mathbf{A}_d\Big]\end{aligned} \tag{41}$$

for U_2, together with

$$U_3^{(1)} = -s_J^a\mathbf{A}_a \tag{42}$$

$$U_3^{(2)} = \tfrac{1}{2}\left(-ys_J^a\mathbf{A}_a + s_J^as_J^b\mathbf{A}_a\mathbf{A}_b\right) \tag{43}$$

$$U_3^{(3)} = \frac{1}{3!}\left(-y^2s_J^a\mathbf{A}_a + (2\bar{y}+y)s_J^as_J^b\mathbf{A}_a\bar{\mathbf{A}}_b - s_J^as_J^bs_J^c\mathbf{A}_a\mathbf{A}_b\mathbf{A}_c\right) \tag{44}$$

$$\begin{aligned}U_3^{(4)} = \frac{1}{4!}\Big[&-y^3s_J^a\mathbf{A}_a + (3\bar{y}^2+3y\bar{y}+y^2)s_J^as_J^b\mathbf{A}_a\bar{\mathbf{A}}_b - (3\bar{\bar{y}}+2\bar{y}+y)s_J^as_J^bs_J^c\mathbf{A}_a\bar{\mathbf{A}}_b\bar{\bar{\mathbf{A}}}_c\\ &+s_J^as_J^bs_J^cs_J^d\mathbf{A}_a\mathbf{A}_b\mathbf{A}_c\mathbf{A}_d\Big]\end{aligned} \tag{45}$$

for U_3. Let us emphasize that in all the expressions above for U_1, U_2 and U_3, the connection is evaluated at the vertex v. The bars only serve to indicate the position at which the corresponding derivative acts.

Next we write the contributions to the holonomy in powers of the segments. According to equations (20) and (30) we obtain

$$\mathbf{h}^{(2)}_{\alpha_{IJ}} = \tfrac{1}{2} s^a_I s^b_J \mathbf{F}_{ab} \tag{46}$$

$$\mathbf{h}^{(3)}_{\alpha_{IJ}} = \frac{1}{3!}\left(s^c_I + s^c_J\right) s^a_I s^b_J \mathbf{D}_c \mathbf{F}_{ab} \tag{47}$$

$$\mathbf{h}^{(4)}_{\alpha_{IJ}} = \frac{1}{4!}\left(s^c_I s^d_I + s^c_I s^d_J + s^c_J s^d_J\right) s^a_I s^b_J \mathbf{D}_c \mathbf{D}_d \mathbf{F}_{ab} + \tfrac{1}{8} s^a_I s^b_J s^c_I s^d_J \mathbf{F}_{ab} \mathbf{F}_{cd} \tag{48}$$

$$\begin{aligned}\mathbf{h}^{(5)}_{\alpha_{IJ}} = \frac{1}{5!}&\left(s^c_I s^d_I s^e_I + s^c_J s^d_J s^e_J + 2\left(s^c_J s^d_I s^e_I + s^c_I s^d_J s^e_J\right) - s^c_I s^d_J s^e_I - s^c_J s^d_I s^e_J\right) s^a_I s^b_J \mathbf{D}_c \mathbf{D}_d \mathbf{D}_e \mathbf{F}_{ab} \\ &+ \frac{1}{4!} s^a_I s^b_J s^d_I s^e_J \left(\mathbf{F}_{ab}\left(s^c_I + s^c_J\right)\mathbf{D}_c \mathbf{F}_{de} + \left(s^c_I + s^c_J\right)(\mathbf{D}_c \mathbf{F}_{de})\mathbf{F}_{ab}\right).\end{aligned} \tag{49}$$

Equation (48) resolves the ordering ambiguity which apparently arose in covariantizing the first term on the RHS of equation (16). Nevertheless, as we subsequently show, there is really no such ambiguity at this order. Let us consider the combination

$$\begin{aligned} s^a_I s^b_J s^c_I s^d_J \left(\mathbf{D}_c \mathbf{D}_d - \mathbf{D}_d \mathbf{D}_c\right) \mathbf{F}_{ab} &= s^a_I s^b_J s^c_I s^d_J \left[\mathbf{D}_c, \mathbf{D}_d\right] \mathbf{F}_{ab} \\ &= -s^a_I s^b_J s^c_I s^d_J \left[\mathbf{F}_{cd}, \mathbf{F}_{ab}\right] = [\mathbf{F}, \mathbf{F}] = 0 \end{aligned} \tag{50}$$

where we have used the notation $\mathbf{F} = s^a_I s^b_J \mathbf{F}_{ab}$ together with the property

$$[\mathbf{D}_c, \mathbf{D}_d]\mathbf{G} = -[\mathbf{F}_{cd}, \mathbf{G}] \tag{51}$$

valid for any object $\mathbf{G}$ in the adjoint representation (Jacobi identity).

Results (46)–(48), which we have obtained by direct calculation, constitute in fact the unique gauge covariant generalization of the corresponding Abelian expressions (14)–(16). This provides a strong support to our method of calculation.

5. The holonomy according to the method of [16]

In this section we calculate the holonomy $\mathbf{h}_{\alpha_{IJ}}$ using the method of [16] adapted for our case of three segments: L_1, L_2, L_3. From equations (3.1), (3.3), (3.4), (3.12) and (3.14), with $\lambda = 1$, of that reference it follows that

$$\mathbf{h}_{\alpha_{IJ}} = \exp(L_3)\exp(L_2)\exp(L_1) = \exp\left(\Sigma + \sum_{n=2} F_n\right) = \exp(H). \tag{52}$$

The basic building blocks are

$$L_1 = s^a_I F(x)\mathbf{A}_a(v) \qquad L_2 = \left(s^a_J - s^a_I\right)F(y-x)\,\mathrm{e}^x \mathbf{A}_a(v) \qquad L_3 = -s^a_J F(y)\mathbf{A}_a(v) \tag{53}$$

where the vertex v generalizes the point (x_1, x_2) in the notation of [16]. The calculational method indicated in equations (52) and (53) is clearly not equivalent to the correct prescription (20): $\mathbf{h}_{\alpha_{IJ}} = U_3 U_2 U_1$, with the U given by equation (26) together with the corresponding extensions that take into account the change of the starting point in the corresponding path.

Let us write here those expressions arising from the method in [16] that we will use in our calculation

$$\Sigma = L_1 + L_2 + L_3 \tag{54}$$

$$2!F_2 = [(L_3 + L_2), L_1] + [L_3, L_2] \tag{55}$$

$$3!F_3 = -[\Sigma, F_2] + [(L_3 + L_2)^2 L_1] + \left[L_3^2 L_2\right] \tag{56}$$

$$4!F_4 = -3![\Sigma, F_3] - [\Sigma^2 F_2] + [(L_3 + L_2)^3 L_1] + \left[L_3^3 L_2\right] \tag{57}$$

with the notation $[A^2 B] = [A, [A, B]]$ and so on.

Expanding each segment in powers of the vectors s^a, whose number is denoted by the superindex, leads to (up to third order)

$$L_1^{(1)} = s_I^c \mathbf{A}_c \qquad L_1^{(2)} = \frac{1}{2!} s_I^a s_I^b \partial_b \mathbf{A}_a \tag{58}$$

$$L_1^{(3)} = \frac{1}{3!} s_I^c s_I^b s_I^a \partial_b \partial_c \mathbf{A}_a \tag{59}$$

$$L_2^{(1)} = \left(s_J^a - s_I^a\right) \mathbf{A}_a \qquad L_2^{(2)} = \frac{1}{2!}\left(s_I^b + s_J^b\right)\left(s_J^a - s_I^a\right) \partial_b \mathbf{A}_a \tag{60}$$

$$L_2^{(3)} = \frac{1}{3!}\left[\left(s_J^b s_J^c + s_J^b s_I^c + s_I^b s_I^c\right)\left(s_J^a - s_I^a\right)\right] \partial_b \partial_c \mathbf{A}_a \tag{61}$$

$$L_3^{(1)} = -s_J^c \mathbf{A}_c \qquad L_3^{(2)} = -\frac{1}{2!} s_J^a s_J^b \partial_b \mathbf{A}_a \tag{62}$$

$$L_3^{(3)} = -\frac{1}{3!} s_J^c s_J^b s_J^a \partial_b \partial_c \mathbf{A}_a. \tag{63}$$

Next we write the contributions to H in powers of the segments. Using equation (52), the first-order contribution $H^{(1)}$ vanishes and the remaining contributions are

$$H^{(2)} = \tfrac{1}{2} s_I^a s_J^b \mathbf{F}_{ab} \tag{64}$$

$$H^{(3)} = \frac{1}{3!}\left(s_I^c + s_J^c\right) s_I^a s_J^b \mathbf{D}_c \mathbf{F}_{ab} + \Sigma^{(3)} \tag{65}$$

with

$$\Sigma^{(3)} = \tfrac{1}{12}\left(-s_I^b s_I^a s_J^c + s_I^b s_J^a s_J^c + s_J^b s_I^a s_J^c + s_J^b s_J^a s_I^c - s_J^b s_I^a s_I^c - s_I^b s_J^a s_I^c\right)[\partial_b \mathbf{A}_a, \mathbf{A}_c]. \tag{66}$$

The reader can verify that the term $\Sigma^{(3)}$ is not covariant under the gauge group. This would not be the case if one had used the path-ordering prescription (that is to say equation (20) instead of equation (52)) in the construction of the holonomies. Indeed, gauge covariance should hold to each order in the expansion in powers of the segments.

6. Summary and discussion

We have calculated the holonomy $\mathbf{h}_{\alpha_{IJ}}$ of the Yang–Mills connection $\mathbf{A}_a$ in the triangle α_{IJ} with vertex v and sides s_I^a, s_J^b joining at that vertex, as shown in figure 1. Our results, to fifth order in the segments, are given in equations (46)–(49) of section 4. The direct calculation shows that, to fourth order, the results are directly given by the replacement $\partial_a \rightarrow \mathbf{D}_a$, $F_{ab} \rightarrow \mathbf{F}_{ab}$ in the corresponding formulae for the Abelian case. This is so because, as explained at the end of section 4, the potential order ambiguity in the fourth-order term is absent. From the fifth order on such ordering ambiguities arise, so that it is not possible to guess the correct answer from the Abelian case. Clearly then, one has to perform the full calculation in order to obtain the correct result.

In section 5 the same calculation was performed using the method of [16]. The results for the triangle coincide up to second order and start to differ from the third order on. This

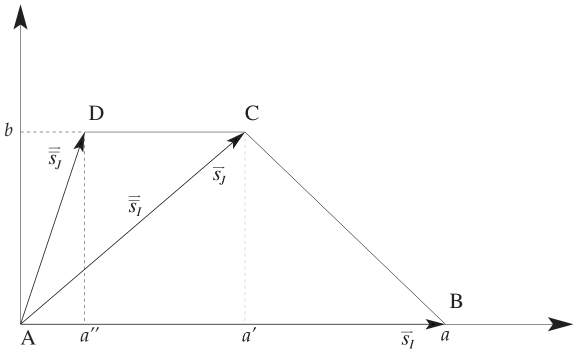

Figure 2. Quadrilateral ABCD with vertex v at the point A.

difference can be traced back to that arising in the calculation of the holonomy for a straight-line segment according to equations (34) and (35), which the reader can easily verify. Contrary to what is expected, the calculation according to the method in [16] produces non-covariant contributions starting at third order.

The specific results presented in [16] correspond to the calculation of the holonomy for a rectangle of sides a and b, respectively. Using the method described in section 4 we have verified that, up to fourth order, the result given in equation (3.19) of that reference is correct.

Nevertheless, since the calculational method of [16] does not properly take into account the path ordering, it is not possible to guarantee the multiplicative composition law of holonomies. In fact, one might think of obtaining the holonomy for the rectangle by composing the results of properly chosen triangles. Even though, as commented above, one should not expect to obtain the correct result, we have explored this possibility. To this end let us consider a quadrilateral ABCD with vertex v at the point A, as shown in figure 2. The sides AB and CD are parallel, but DA and BC are not. We are interested in calculating the holonomy for the path ABCDA as indicated in figure 2. This can be done by composing, via matrix multiplication, the holonomies corresponding to the triangles ABC (spanned by the vectors $\vec{s}_I, \vec{s}_J$) and ACD (spanned by the vectors $\bar{\vec{s}}_I, \bar{\vec{s}}_J$), each of which is calculated according to expressions (65) and (66), together with their corresponding Σ. In other words,

$$\mathbf{h}_{\mathrm{ABCDA}} = \mathbf{h}_{\mathrm{ACDA}} \mathbf{h}_{\mathrm{ABCA}}. \tag{67}$$

We introduce further notation in the plane of the quadrilateral

$$\vec{\mathrm{AB}} = \vec{s}_I = (a, 0) \qquad \vec{\mathrm{AC}} = \vec{s}_J = (a', b) \tag{68}$$

$$\vec{\mathrm{AC}} = \bar{\vec{s}}_I = (a', b) \qquad \vec{\mathrm{AD}} = \bar{\vec{s}}_J = (a'', b). \tag{69}$$

We only pay attention to the non-covariant contributions. To third order they just add up and we obtain

$$\begin{aligned}\Sigma^{(3)}_{\mathrm{ABCA}} + \Sigma^{(3)}_{\mathrm{ACDA}} = {} & \frac{(a'' - a' + a)}{12} \{3a'(a'' - a)[\partial_1 \mathbf{A}_1, \mathbf{A}_1] \\ & + b(a'' + a' - a)([\partial_1 \mathbf{A}_1, \mathbf{A}_2] + [\partial_1 \mathbf{A}_2, \mathbf{A}_1] + [\partial_2 \mathbf{A}_1, \mathbf{A}_1]) \\ & + b^2[[\partial_2 \mathbf{A}_1, \mathbf{A}_2] + [\partial_2 \mathbf{A}_2, \mathbf{A}_1] + [\partial_1 \mathbf{A}_2, \mathbf{A}_2]]\}.\end{aligned} \tag{70}$$

We see that the above expression is not zero in general. Nevertheless, in the symmetrical case of a parallelepiped, characterized by the condition

$$a'' - a' + a = 0 \tag{71}$$

the non-covariant piece (71) vanishes. Of course, the above condition includes that of a rectangle which is $a = a'$ and $a'' = 0$.

Moreover, to fourth order in the expansion, it is possible to show that following the calculational method of [16], the composition law (67) yields gauge covariance violations even for the case of the parallelepiped.

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